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The Wigner–Lubanski First Integral in the Papapetrou–Corinaldesi Equations of Motion[1]

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Abstract

Some years ago Papapetrou and Corinaldesi applied Papapetrou's equations of motion of spinning particles to the case of motion in the Schwarzschild field. For the particular case of motion in the equatorial plane they found an extra integral of motion (in addition to the constants of energy and total angular momentum). We here give a group-theoretical interpretation to the origin of this constant by relating it to the Wigner–Lubanski constant known from the theory of representations of the Poincaré group.

Some years ago Papapetrou developed equations of motion for spinning particles that later were applied to motion in the Schwarzschild field using the supplementary condition $S^{i0} = 0$, where $S^{\mu\nu}$ describes the spin of the particle (Papapetrou, 1951; Corinaldesi and Papapetrou, 1951). The equations of motion were subsequently integrated and first integrals (constants of motion) were obtained corresponding to the total angular momentum **J** and the energy E. Apart from some special cases the general condition for a motion in the equatorial plane $\theta = 0$ was found to be $S^{23} = S^{12} = 0$, with only $S^{31} \neq 0$. We here use Papapetrou's original notation (except for denoting x^4 by x^0) according to which $\theta = 0$ describes the equatorial plane rather than $\theta = \pi/2$, as is usually done. The physical meaning of the above conditions is that the Cartesian components of the spin are given by $S_x = S_y = 0$, and $S_z = rS^{13}$. In this particular case it was found that there is an additional integral of motion, denoted by F.

Nobody has yet found a group-theoretical explanation for the origin of this extra integral of motion. In this paper we relate this extra integral of motion to a generalization into curved space of the Wigner–Lubanski invariant, which is one of the Casimir operators that occur in the theory of representa-

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tions of the Poincaré group and which characterizes the representation of the little group (Wigner, 1939, 1963). Its meaning in that case is the square of the total angular momentum in the coordinate system in which the particle is at rest, multiplied by the square of the mass. More specifically we show below that the product of the constants E and F is related to a Wigner–Lubanski type of constant W by $W = \tilde{F}^2$, where $\tilde{F} = EF$. The explicit forms of E and $\tilde{F}$ are (Corinaldesi and Papapetrou, 1951)

$$E = e^{\mu}\dot{t}(m + m_s) = \text{const} \tag{1}$$

$$\tilde{F} = r\dot{t}(m + m_s)S^{31} = \text{const} \tag{2}$$

Here $(m + m_s)$ is an *effective* mass of the spinning particle, and a dot denotes differentiation with respect to the proper time s. The function e^{μ} is related to the Schwarzschild metric by $ds^2 = e^{\mu}\, dt^2 - e^{-\mu}\, dr^2 - r^2(d\theta^2 + \cos^2\theta\, d\phi^2)$.

A generalized Wigner–Lubanski constant is now defined by

$$W = -g^{\mu\nu}w_\mu w_\nu \tag{3}$$

where

$$w_\mu = {}^*S_{\mu\alpha}p^\alpha \tag{4}$$

and ${}^*S_{\mu\nu}$ is the *dual* to the spin tensor $S^{\alpha\beta}$,

$${}^*S_{\mu\nu} = \tfrac{1}{2}(-g)^{1/2}\epsilon_{\mu\nu\alpha\beta}S^{\alpha\beta} \tag{5}$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol with $\epsilon_{0123} = 1$, and p^α is the total linear momentum (kinetic plus spin contribution),

$$p^\mu = mu^\mu + u_\beta DS^{\mu\beta}/Ds \tag{6}$$

where $DS^{\mu\beta}/Ds = S^{\mu\beta}_{;\alpha}u^\alpha$, a semicolon denotes covariant differentiation, and $u^\mu = dx^\mu/ds$.

In the particular case of motion in the equatorial plane $\theta = 0$ with only one nonvanishing component of spin $S^{13} \neq 0$ one obtains from equation (4)

$$\begin{aligned} w_0 &= {}^*S_{02}p^2 \\ w_2 &= {}^*S_{20}p^0 \end{aligned} \tag{7}$$

whereas $w_1 = w_3 = 0$, since the only nonvanishing component of the dual spin tensor is ${}^*S_{02} = -{}^*S_{20}$, which is given in the present case by

$${}^*S_{02} = -r^2S^{13} \tag{8}$$

Therefore one obtains

$$W = -[g^{00}(w_0)^2 + g^{22}(w_2)^2] \tag{9}$$

A straightforward calculation then leads to the following expressions for p^0 and p^2:

$$\begin{aligned} p^0 &= \dot{t}(m + m_s) \\ p^2 &= 0 \end{aligned} \tag{10}$$

Accordingly

$$\begin{aligned} W &= -g^{22}(w_2)^2 \\ &= [rS^{31}\dot{t}(m + m_s)]^2 \\ &= \tilde{F}^2 \end{aligned} \tag{11}$$

thus proving the statement made above.

It is interesting to note that the Wigner–Lubanski constant of motion has a meaning that is independent of the energy in the case of the spinning particle's motion. It can be shown that in the case of Vaidya's radiating Schwarzschild metric (Vaidya, 1943, 1951, 1953) the energy of the spinning particle is not conserved. In spite of that, the Wigner–Lubanski extra integral of motion W is still a constant of the motion in the particular case of motion in the plane (Carmeli et al., 1977).

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